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Procedia Structural Integrity 35 (2022) 2–9

Structural Integrity Procedia

www.elsevier.com/locate/procedia

2nd International Workshop on Plasticity, Damage and Fracture of Engineering Materials

Yield criteria representable by elliptic curves and Weierstrass form

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Abstract

Elliptic curve terminology comes from the close association with elliptic functions, and not because of any physical resemblance to an ellipse. The curves investigated here represent various yield loci in the plane having cubic algebraic relationships between the second and third invariants of the deviatoric stress tensor. A well-known yield condition attributed to Drucker falls into this classification. In addition, the more commonly used Tresca yield condition represents a limiting case of elliptic curves. All yield criteria based on elliptic curves, including the Tresca, can be parameterized in terms of the Weierstrass elliptic $\wp$-function. The properties of elliptic curves as they pertain to the formulation of various plastic yield criteria of materials are the topic of this investigation. Various perfectly plastic solutions of mode I crack problems are discussed.

Keywords: Elliptic curves; transition model Tresca to von Mises yield condition; plane stress problems; perfectly plastic mode I crack solutions

1. Introduction

Elliptic curves are generated from cubic algebraic equations of a specific form that have special group properties, Solovyov (1999). Because of these special characteristics, they have been important historically in the field of number theory, Knapp (1992), McKean and Moll (1999), and are believed to be key in solving the Birch and Swinnerton-Dyer Conjecture, Ash and Gross (2012), Stewart (2013). Proof of this particular conjecture is considered one of the outstanding problems in all of mathematics, Devlin (2002). Furthermore, elliptic curves have found application in the

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10.1016/j.prostr.2021.12.041

field of cryptography, Washington (2003), and through their connection with modular forms were instrumental in the solution of Fermat's Last Theorem, Devlin (2002). Being the simplest class of algebraic curves beyond conics and lines, Kendig (2011), they can also be found in many other areas of science and engineering. For example, it will be shown here that two commonly used material yield criteria, the Tresca, see Chakrabarty (1987), and the Drucker (1949, 1962), can be reduced to Weierstrass form, a class of functions to which elliptic curves belong, Knapp (1992). This form generates elliptic curves as well as related cubic curves, which are not considered elliptic due to the presence of singularities, Washington (2003). For example, the semicubical parabola, while having an equation of Weierstrass form, is not considered elliptic as it exhibits a cusp. Similarly, the alpha curve, Kendig (2011), which is also generated by an equation of Weierstrass form, is not considered elliptic, because of the appearance of a node. The Tresca yield condition has a similar shape to the alpha curve when expressed in terms of its (J_2, J_3) invariants of the deviatoric stress tensor in Weierstrass form (*X*, *Y*). However, a slight perturbation of the Tresca yield condition will generate a yield locus that is a true elliptic curve. The yield condition of Drucker is also an elliptic curve when expressed in terms of the same deviatoric invariants (J_2, J_3) in Weierstrass form (*X*, *Y*). The only other class of singularity found in cubic equations besides nodes and cusps are isolated points, Bix (2006).

Note that yield criteria that have hydrostatic stress dependence, such as the Mohr-Coulomb or Drucker-Prager, see Chen and Zhang (1991), are not addressed here as they cannot be expressed in terms of (J_2, J_3) alone.

A form of cubic equation, which admits several different yield conditions as special cases, has the representation

$$(J_3 / k^3)^2 = \alpha (J_2 / k^2)^3 + \beta (J_2 / k^2)^2 + \gamma J_2 / k^2 + \delta, \tag{1}$$

where J_2 and J_3 are the second and third invariants of the deviatoric stress tensor, Chakrabarty (1987), where the Greek symbols represent constants, and where *k* is the yield strength in pure shear. In this analysis, the deviatoric stress invariants will be restricted to those cases where the third principal stress σ_3 is zero, as for plane stress problems. It follows that

$$J_2 = \frac{1}{3}\left(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right), \tag{2}$$

$$J_3 = \frac{\sigma_1 + \sigma_2}{27}\left(2\sigma_1^2 - 5\sigma_1\sigma_2 + 2\sigma_2^2\right), \tag{3}$$

where σ_1 and σ_2 are the first and second principal stresses respectively. The Weierstrass form of (1) is expressible in terms of Cartesian coordinates (*X*, *Y*) through the following substitutions

$$Y^2 = X^3 + c_1 X + c_2, \quad \text{where} \tag{4}$$

$$X = \frac{J_2}{k^2} + \frac{1}{3}\frac{\beta}{\alpha}, \quad Y = \frac{J_3}{\sqrt{\alpha} k^3}, \quad \text{and} \tag{5}$$

$$c_1 = \frac{\gamma}{\alpha} - \frac{1}{3}\frac{\beta^2}{\alpha^2}, \quad c_2 = \frac{2}{27}\frac{\beta^3}{\alpha^3} - \frac{1}{3}\frac{\beta\gamma}{\alpha^2} + \frac{\delta}{\alpha}. \tag{6}$$

Table 1. Cubic equations and their relationships to yield criteria through deviatoric stress invariants.

Yield Condition	X	Y	k	Cubic Equation
Tresca	$\frac{J_2}{k^2}-3$	$\frac{3\sqrt{3}}{2}\frac{J_3}{k^3}$	$0.500\,\sigma_0$	$Y^2 = X^3 - 3X + 2$
Drucker	$\frac{J_2}{k^2}$	$\frac{3}{2}\frac{J_3}{k^3}$	$0.540\,\sigma_0$	$Y^2 = X^3 - 1$
E1	$\frac{J_2}{k^2}-2$	$\frac{J_3}{k^3}$	$0.542\,\sigma_0$	$Y^2 = X^3 - X$
E2	$\frac{J_2}{k^2}-3$	$\frac{J_3}{k^3}$	$0.573\,\sigma_0$	$Y^2 = X^3 - X + 6$
von Mises	$\frac{J_2}{k^2}$	-	$0.577\,\sigma_0$	$X = 1$

In Table 1, one finds four different yield conditions representable in the form of (4), plus the von Mises yield condition, Chakrabarty (1987), for comparison. The yield criteria tabulated as E1 and E2 are elliptic curves whose properties are investigated by the author. Their priority remains unknown to the author, but because of their simplicity, the author assumes none. In Table 1, the symbol σ_0 represents the yield strength in tension.

In Fig. 1 (a), the various yield conditions presented in Table 1 are plotted in the *XY* plane. In Fig. 1 (b), the

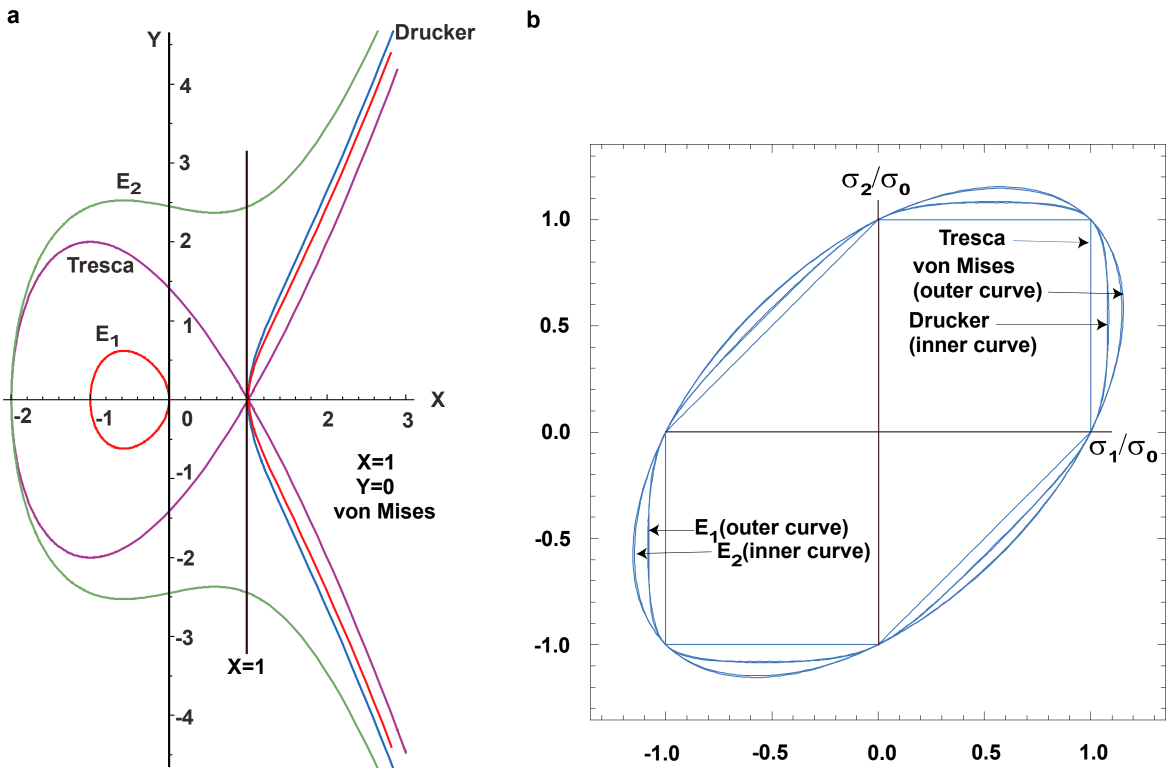

Fig. 1. (a) various yield conditions in the *XY* plane; (b) yield conditions in the normalized principal stress plane.

corresponding yield criteria are plotted in the normalized principal stress plane. Note that the von Mises yield condition and yield condition E2 are nearly indistinguishable when plotted in the principal stress plane. Similarly, the Drucker yield condition and yield condition E1 are also very close to one another in this plane.

2. Parameterization of Weierstrass form

One interesting feature of yield conditions of the form (1) is that they are expressible parametrically in terms of the Weierstrass $\wp$-functions. This is evident by examining the following ordinary differential equation associated with this function, Ambramowitz and Stegun (1964),

$$\left[\wp'(u)\right]^2 = 4\wp(u)^3 - g_2\wp(u) - g_3, \tag{7}$$

where the prime on $\wp(u)$ designates differentiation with respect to the independent variable *u*, which serves as a parameter. The Weierstrass function, its first derivative, and the relationships that *X* and *Y* assume with respect to this function's invariants (g_2, g_3) are

$$\wp(u) = X, \quad \wp'(u) = 2Y, \quad g_2 = -4c_1, \quad g_3 = -4c_2, \tag{8}$$

where the constants c_1 and c_2, defined previously in (6), are provided for individual yield conditions in Table 2.

Table 2. Coefficients of the various yield conditions as they relate to Weierstrass form.

Yield Condition	α	β	γ	δ	c_1	c_2
Tresca	4/27	-4/3	32/9	-64/27	-3	2
Drucker	4/9	0	0	-4/9	0	-1
E1	1	-6	11	-6	-1	0
E2	1	-9	26	-18	-1	6
von Mises, $X = 1$	N/A	N/A	N/A	N/A	N/A	N/A

Note that the Weierstrass $\wp$-function is reducible to alternative forms in specific cases, Ambramowitz and Stegun (1964), Gradshteyn and Ryzhik (1980). For example, for the various yield conditions based on cubic equations presented in Tables 1 and 2, one finds

$$\text{Tresca:} \quad \wp(u) = 3\coth^2\left(\sqrt{3}\,u\right) - 2, \tag{9}$$

$$\text{Drucker:} \quad \wp(u) = 1 + \sqrt{3}\,\frac{1 + \mathrm{cn}\left(2\cdot 3^{1/4}u, m\right)}{1 - \mathrm{cn}\left(2\cdot 3^{1/4}u, m\right)}, \text{ where } m = \frac{1}{2} - \frac{\sqrt{3}}{4}, \tag{10}$$

$$\text{E1:} \quad \wp(u) = \frac{2}{\mathrm{sn}^2\left(\sqrt{2}\,u, 1/2\right)} - 1, \tag{11}$$

$$\text{E2:}\quad \wp(u) = -2 + \sqrt{11}\frac{1+\text{cn}\left(2\cdot 11^{1/4}u, m\right)}{1-\text{cn}\left(2\cdot 11^{1/4}u, m\right)},\ \text{where } m = \frac{1}{2} + \frac{3}{2\sqrt{11}}. \tag{12}$$

In (10)-(12), the function cn and sn are Jacobian elliptic functions as defined in Ambramowitz and Stegun (1964).

3. Tresca yield condition and its generalization

In solving the fundamental mode I crack problem for the Drucker perfectly plastic yield condition, Unger (2008, 2009) defined the following stress function $\phi(r,\theta)$ for use in polar coordinates (r,θ)

$$\begin{aligned}&\phi(r,\theta) = r^2 f(\theta), \quad p = f'(\theta),\\ &\sigma_\theta = 2f(\theta), \quad \tau_{r\theta} = -f'(\theta) = -p, \quad \sigma_r = f''(\theta) + 2f(\theta) = p\frac{dp}{df} + 2f,\end{aligned} \tag{13}$$

where σ_θ and σ_r are normal stresses, $\tau_{r\theta}$ is the shear stress, and the primes denote differentiation with respect to θ. For a plane stress problem, involving the Tresca yield condition, the yield condition assumes the following form in terms of the stress function *f* and its derivatives

$$\begin{aligned}&\left(2k^2 + kq + fq - Q\right)\left(2k^2 - kq + fq - Q\right)\left(4k^2 - q^2 + 8\left(fq - Q\right)\right) = 0,\\ &\text{where } Q = p^2/2 + 2f^2, \quad q = dQ/df.\end{aligned} \tag{14}$$

The general solution of the nonlinear ordinary differential equation defined in (14) follows

$$\begin{aligned}&f = -k + \left(C_1/2\right)\left(1 + \cos\left(2\theta + C_2\right)\right), \quad 0 \le C_1 \le k,\\ &f = k + \left(C_3/2\right)\left(1 + \cos\left(2\theta + C_4\right)\right), \quad -k \le C_3 \le 0,\\ &f = C_5 + \left(k/2\right)\sin\left(2\theta + C_6\right), \quad -k \le 2C_5 \le k,\end{aligned} \tag{15}$$

where $C_i\ (i = 1, \ldots, 6)$ are constants to be determined from boundary conditions on traction.

The phase plane of this solution (15) is depicted in Fig. 2 (a), where each specific form of *f* in (15) generates a family of ellipses upon variation of the parameters with the odd-valued subscripts between the limits indicated. By inspection of Fig. 2 (a), candidates for singular solutions of (14) also exist at $p = \pm k$, because singular solutions correspond to envelopes of the general solution loci in the phase plane. Note that singular solutions cannot be obtained from the general solution by simply selecting particular values of the arbitrary constants. Upon integration of $p = \pm k$, one finds

$$f = \pm k\theta + C_7,\ \text{where} \left|f/k\right| \le 1/2. \tag{16}$$

By direct substitution of (16) into (14), it is verified that they constitute singular solutions of (14).

An extension of the Tresca yield condition is now derived from the Weierstrass form provided in Table 1 by simply appending a constant ε to its end, i.e.,

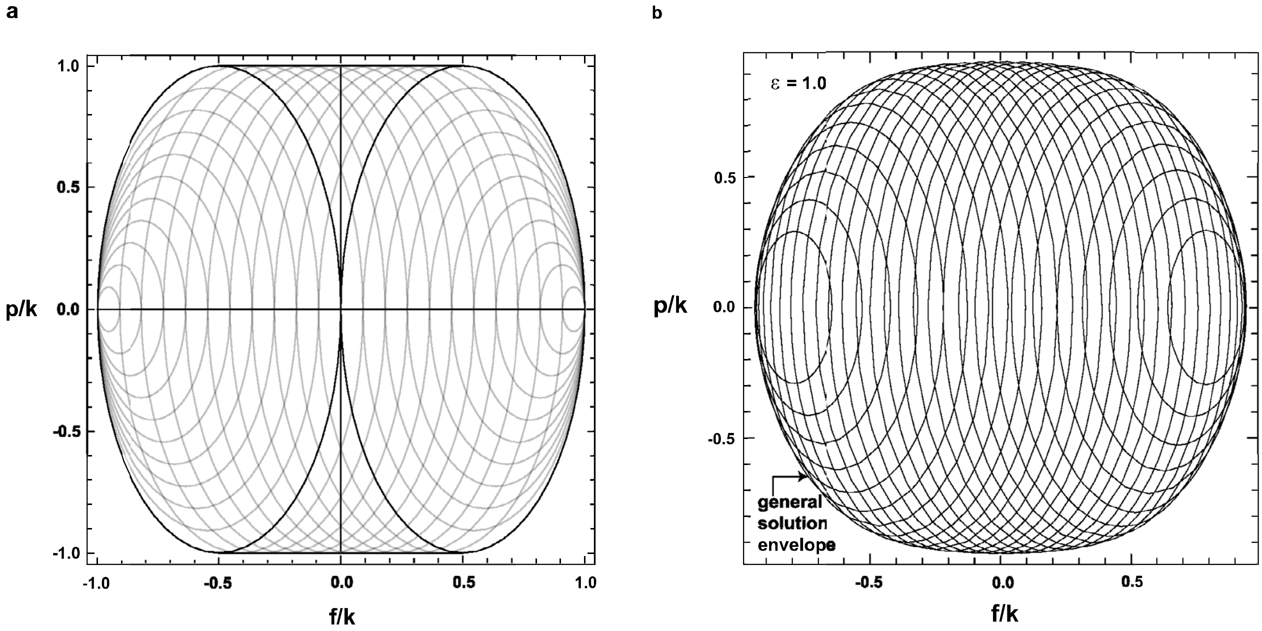

Fig. 2. (a) phase plane of solution of Tresca yield condition; (b) typical phase plane of solution of generalized Tresca yield condition.

$$Y^2 = X^3 - 3X + 2 + \varepsilon, \quad 0 \le \varepsilon < 16. \tag{17}$$

The coordinates X and Y remain the same as in Table 1 in terms of the deviatoric stress invariants; however, the parameter k can no longer be interpreted as the yield strength in pure shear. By choosing ε appropriately, a yield condition is generated that allows experimental values for yield strengths in both tension σ_0 and in pure shear τ_0 to be independently incorporated into it, i.e.,

$$\varepsilon = \left(4 - \left(\frac{\sigma_0}{k}\right)^2\right)^2 = \left(1 - \left(\frac{\tau_0}{k}\right)^2\right)\left(4 - \left(\frac{\tau_0}{k}\right)^2\right)^2 \text{ or in inverted form} \tag{18}$$

$$\sigma_0 = k\sqrt{4 - \sqrt{\varepsilon}}, \quad \tau_0 = k\sqrt{1 - 4\sinh^2\left[(1/3)\sinh^{-1}\left(\sqrt{\varepsilon}/2\right)\right]}, \tag{19}$$

where a solution of a cubic algebraic equation was used to obtain (19) from (18), Weisstein (2002). Utilizing (19), one obtains the generalized Tresca yield condition in terms of principal stresses as

$$\begin{aligned} &4\left(\sqrt{\varepsilon} - 4\right)\sigma_0^2\left(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right)^2 + 32\sigma_0^4\left(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right) \\ &+\left(\sqrt{\varepsilon} - 4\right)^2\left(\sigma_1 - \sigma_2\right)^2\sigma_1^2\sigma_2^2 - 4\left(\sqrt{\varepsilon} + 4\right)\sigma_0^6 = 0. \end{aligned} \tag{20}$$

Note that in the limit as $\varepsilon \to 16$ the von Mises yield condition is recovered from (20) for plane stress loading conditions. Thus, a continuous transformation occurs $\left(0 \le \varepsilon \le 16\right)$ from the Tresca to von Mises yield conditions.

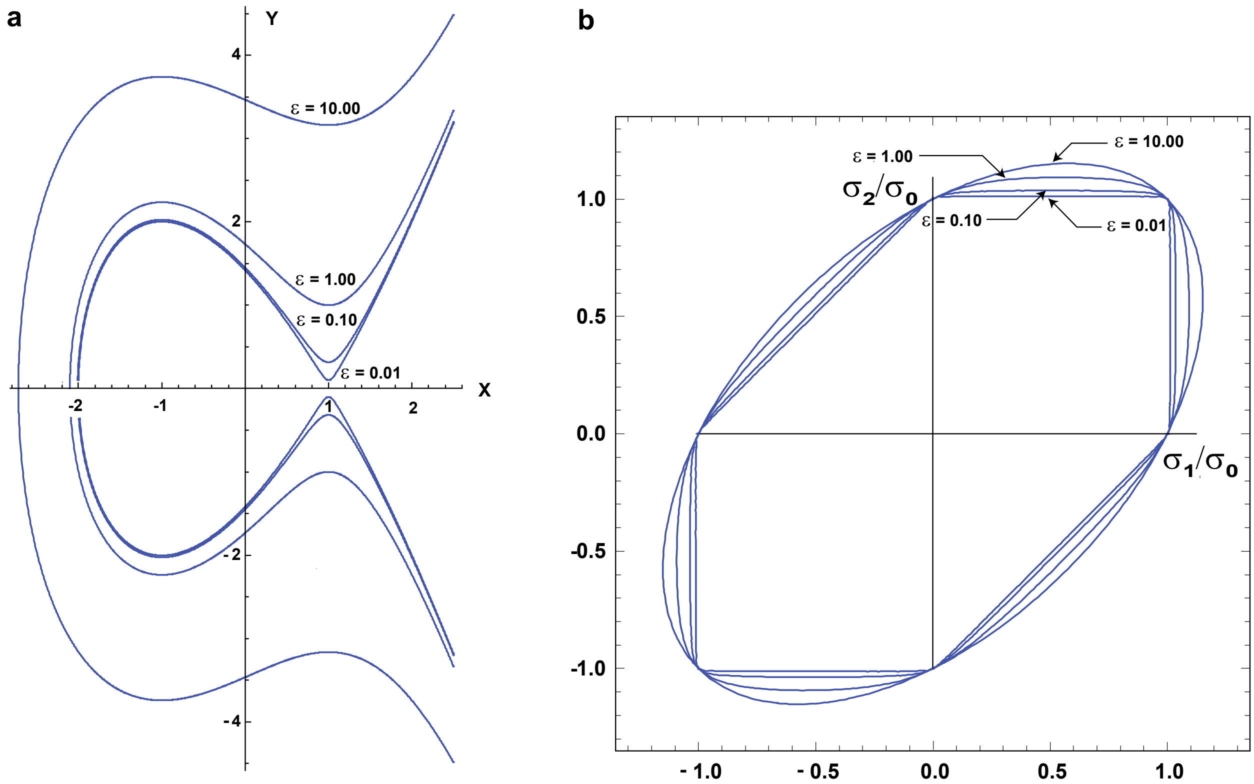

Fig. 3. (a) Elliptic curves of generalized Tresca yield condition; (b) generalized Tresca yield condition in normalized principal stress plane.

In Fig. 2 (b), the phase plane of solutions of the generalized Tresca yield condition is displayed for the case ε equals one. Unlike the conventional Tresca yield condition, which is factorable into three terms (14), there is but one specific form of the general solution, instead of three distinct cases (15). For details on how to obtain such a solution, see Unger (2008, 2009). Furthermore, the candidate for a singular solution in this case circumscribes the entire locus of the general solution, unlike the conventional Tresca yield condition whose branches of the singular solution are restricted to the flat envelopes that are located at the top and bottom portions of the phase plane.

Details of an analytical solution of the mode I crack problem for a perfectly plastic material employing the Drucker yield condition under plane stress loading conditions are provided in Unger (2008, 2009).

4. Discussion

It is interesting to note the large class of yield criteria that are expressible in Weierstrass form. Because of the intrinsic relationship between this form and the Weierstrass elliptic $\wp$-function, a novel parameterization of yield condition is possible between this function and its first derivative. The Weierstrass $\wp$-function can also be expressed in terms of Jacobian elliptic functions, Ambramowitz and Stegun (1964), and in the special case of the Tresca yield condition, an elementary transcendental function (9), Gradshteyn and Ryzhik (1980). The Tresca yield condition assumes a particularly simple form when expressed in the cubic curve representation (*X*, *Y*). From this simple representation, the addition of a constant ε appended to it allows a form of yield condition from which one can incorporate different experimental values of yield strengths in tension and shear (18), (19). Also, by varying this parameter ε, one can obtain a family of yield criteria that superficially resemble other yield conditions derived from other forms of cubic equations in the principal stress plane. This is observable by comparing Fig. 1 (b) and Fig. 3 (b). For small values of ε, a smooth approximation of the Tresca yield condition is obtained as shown in Fig. 3 (b) for $\varepsilon = 0.01$.

Singular solutions are often employed in the solution of mode I crack problems for perfectly plastic materials. In Unger (2008, 2009), two different plane stress, perfectly plastic solutions were explored for the governing differential equations in the phase plane. One was for the von Mises yield condition, whose solution was first obtained by Hutchinson (1968), and the other was for the Drucker yield condition. Singular solutions occupy the leading sector ahead of the crack tip in both these solutions. One notes how the loci of the general solution for the Tresca and its generalization differ dramatically regarding the formation of envelopes. At present, no completely satisfactory mode I crack solution for a perfectly plastic material has been found for the traditional Tresca yield condition under plane stress loading conditions, although an effort was made toward this goal in Unger (2005). In contrast, it is entirely possible that one exists for the generalized Tresca yield condition, as its envelope for ε equals one resembles that of the Drucker yield condition for which a suitable analytical solution exists. This is a topic for future study and analysis.

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